MEASURING SHIFTS IN REASONING ABOUT FRACTION ARITHMETIC IN A MIDDLE GRADES NUMBER AND OPERATIONS CONTENT COURSE

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The present study extends recent advances developing and applying measures of mathematical content knowledge for teaching. Recent research has demonstrated that the Diagnosing Teachers' Multiplicative Reasoning Fractions survey provides information about distinct but related components necessary for reasoning in terms of quantities when solving fraction arithmetic problems. The present study adds a new component of validity for the survey by examining the extent to which one pre-service teacher's growth in reasoning about fraction arithmetic, as indicated by assessments she completed for a middle grades numbers and operations course, was reflected in her performance on the survey. Results provide an existence proof that the survey is sensitive to shifts towards more proficient reasoning.

Keywords: Mathematical Knowledge for Teaching, Measurement, Teacher Education-Preservice

Recent curriculum standards documents (e.g., National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010; National Council of Teachers of Mathematics, 2000) and recommendations for teacher education (e.g., American Mathematical Society, 2010; Sowder et al., 1998) have placed high value on developing conceptual understanding by solving and reflecting on solutions to problems embedded in situations. These standards and recommendations have led to two critical challenges for mathematics teacher education addressed by the present study. First, a body of past research (e.g., Ball, 1990; Izsák, 2008; Ma, 1999) has reported persistent difficulties preservice and inservice teachers have had explaining fraction arithmetic in terms of quantities and has demonstrated a need for teacher education to attend to this critical content. Second, the field has been striving to develop measures that target mathematical knowledge needed for teaching practice (e.g., Bradshaw, Izsák, Templin, & Jacobson, 2014; Hill, 2007; Izsák, Jacobson, de Araujo, & Orrill, 2012; Kersting, Givvin, Sotelo, & Stigler, 2010). For us, addressing both challenges simultaneously means developing and utilizing measures that capture information about teachers' capacities to construct chains of reasoning for explaining fraction arithmetic in terms of quantities.

The present study addresses both challenges identified above by examining the first application of the Diagnosing Teachers' Multiplicative Reasoning (DTMR) Fractions survey to study growth and change in a cohort of preservice teachers. The important feature distinguishing the DTMR Fractions survey from the Learning Mathematics for Teaching (LMT) measures of mathematical knowledge for teaching (e.g., Hill, 2007) is that it is multi-dimensional. The LMT measures are used to estimate a single score on a continuous scale that can be interpreted as an overall measure of ability with respect to the targeted mathematical content. In contrast, the DTMR survey provides information about four components of reasoning (illustrated with an example below) that are well established in the theoretical and empirical research literature on children's and teachers' reasoning about fraction arithmetic in terms of quantities. The trade-off for multidimensionality is that the DTMR components are dichotomous variables. By measuring components of reasoning, the DTMR survey has potential to capture nuanced information about teachers' reasoning and ways their reasoning can shift during content courses.

In prior work, Bradshaw et al. (2014) reported on the development of the DTMR Fractions survey and analyzed data from a national sample of 990 in-service middle grades teachers. The

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results established the content validity and psychometric properties of the survey. In the present study, we administered the DTMR Fractions survey before and after a semester-long course on number and operations that gave significant attention to reasoning about fraction arithmetic in terms of quantities. The course was offered to 22 preservice mathematics teachers in a middle grades program. The goal of the study was not to evaluate the effectiveness of the course but rather to examine a form of convergent validity around growth and change in the preservice teachers' reasoning. In particular, our research question asked whether or not the reasoning that preservice teachers demonstrated on course assessments was consistent with shifts in their reasoning as indicated by their performance on the DTMR Fractions survey. Thus, this study was a test case examining the extent to which it is possible to measure shifts in multiple components of teachers' reasoning using psychometric methods.

Measuring Teachers' Reasoning About Fraction Arithmetic in Terms of Quantities

As indicated above, solving fraction arithmetic problems in terms of quantities involves multiple components of reasoning. The solution outlined to the following problem illustrates three of the four components measured by the DTMR Fractions survey.

> *A batch of brittle calls for $^1/_4$ of a cup of honey. Megan has $^2/_3$ of a cup of honey. How many batches of brittle can Megan make?*

The solution presented below presumes that a teacher will not employ a general numeric method, such as multiplying by the reciprocal of the divisor, but instead will reason directly with the quantities of cups and batches to solve the problem.

First, to see the opportunity for discussing division, a teacher would have to recognize that the brittle problem asks a how-many-groups question, the signature for measurement division. We use the term *appropriateness* to refer to selecting an arithmetic operation that can model a given problem situation. Next, the teacher might produce a drawn model. The double number line shown in Figure 1a, where lengths depict cups of honey, is one possibility. Our next point about partitioning is not dependent on this choice of drawn model. Juxtaposing the two number lines highlights the challenge that fourths and thirds do not subdivide one another evenly. Whole-number factor-product relationships are useful for overcoming this challenge. In this case, 12 is a common multiple of 3 and 4. Figure 1b illustrates how twelfths simultaneously subdivide thirds and fourths of cups and thus provide a finer unit with which to compare the two. Thus, teachers must be skillful at *partitioning* quantities, often using factor-product combinations as a tool. A related aspect involves *iterating* the resulting mini-piece (8 times in this example).

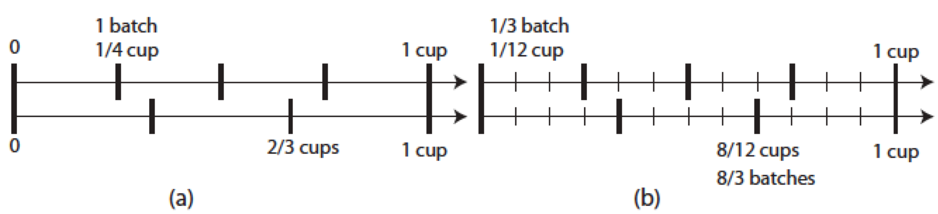

Figure 1. Reasoning with a double number line.

Finally, the teacher must interpret the mini-pieces in terms of the given situation. There are multiple candidates, including interpreting one mini-piece as a twelfth of 1 cup, as a fourth of $^1/_3$ cup, and as a third of $^1/_4$ cup. Because the problem asks about the number of $^1/_4$ cups in $^2/_3$ cups, $^1/_4$ cup is the appropriate referent unit: There are $^8/_3$ $^1/_4$-cups in $^2/_3$ cups. Thus, the teacher must be clear about the *referent units* for all numbers used in the solution. The DTMR Fractions survey provides information about teachers' facility with appropriateness, partitioning and iterating, referent units, and a fourth component, reversibility, that is important for solving partitive division problems. Reversibility has to do with constructing the one whole given a fractional amount of the whole (e.g., $^{3}/_{5}$ or $^{7}/_{5}$).

The DTMR Fractions survey consists of 27 items that measure the four components of reasoning just discussed. We used a psychometric model called the log-linear cognitive diagnosis model (LCDM) to analyze the item responses and estimate profiles that indicate "mastery" or "non-mastery" of appropriateness, partitioning and iterating, referent units, and reversibility. (The term *mastery* comes from the psychometric literature, which we take as a synonym for *proficiency*.) The LCDM is one member of a recently developed family of psychometric models referred to as diagnostic classification models. The DTMR Fractions survey is one of the first practical applications of these new models.

Methods

Data for this report comes from an on-going study of preservice teachers' reasoning about multiplication and division, fractions, and proportional relationships. As part of the broader study, the project team administered the DTMR Fractions survey to a cohort of 22 preservice middle-grades mathematics teachers before and after a number and operations content course offered in Fall 2014. The course was offered as part of a teacher education program at a large, public university in the Southeast United States. The course emphasized reasoning with quantities to develop conceptual understanding of multiplication and division with whole numbers and with fractions. A project-team member taught the course. We administered the DTMR Fractions survey the first week of the course (August, 2014) and again at the beginning of the following semester (January, 2015). In addition to DTMR survey data, we collected all course assessments (quizzes and tests) from all preservice teachers in the course. The preservice teachers completed these assessments between mid-September and mid-December.

We selected six preservice teachers for more detailed study based on their initial fractions profiles as determined by the LCDM analysis. Of the six, the LCDM analysis indicated that Kelly's (a pseudonym) profile shifted along the most (3) dimensions. Thus, we selected her as an initial case to examine the extent to which her growth in reasoning about fraction arithmetic, as indicated by her assessments completed for the number and operations course, was consistent with her shift in profile. The first author first compiled a list of all assessment problems that provided opportunities for Kelly to use at least one of the four components of reasoning—appropriateness, partitioning and iterating, referent units, and reversibility. The compiled problems came from two quizzes, three tests, and the final exam. The first and third authors then individually analyzed Kelly's work on the selected problems, listing evidence for or against facility with each component. The first and third authors then looked at the problems together and compared their lists of evidence for or against facility with each component. Discrepant interpretations were discussed until resolved. The second author then confirmed the first and third authors' analysis, discussing discrepant interpretations with the first and third authors. We then compared Kelly's reasoning as evidenced in her course assessments with the shifts indicated by her performance the DTMR Fractions survey before and after the course.

Results

According to the LCDM analysis, Kelly was not a master of any of the four DTMR components at pretest. Her probabilities of mastery for each component at the beginning of the number and operations course were appropriateness (.01), partitioning and iterating (.01), referent units (.00), and reversibility (.15). At posttest, each of these probabilities increased: appropriateness (.98), partitioning and iterating (.30), referent units (.52), and reversibility (.86). This suggests that by the end of the course Kelly had made significant gains on appropriateness and reversibility, that there was conflicting information about her facility with referent units, and that she still struggled with

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partitioning and iterating. Our analysis indicates that shifts in Kelly's performance from DTMR pretest to DTMR posttest was largely consistent with her performance on the assessments she completed for the numbers and operations course, and that places where her performance diverged corresponded to places where the DTMR Fractions survey was less well aligned with the course assessments.

Shifts in Kelly's Reasoning about Appropriateness

Kelly's performance on assessments for the numbers and operations course indicated that she used tools developed in the course to make appropriate determinations about when and where multiplication and division could be used to model problem situations. These tools were an explicit meaning for multiplication and a form of ratio table. Oftentimes Kelly was explicit about using these two tools. Figure 2 shows one example of her work on a test given in December, near the end of the course. The question presented four word problems and asked whether each one could be modeled by $^{2}/_{3} \div {}^{3}/_{4}$, $^{3}/_{4} \div {}^{2}/_{3}$, or neither. Kelly identified the appropriate operation in all but the second example.

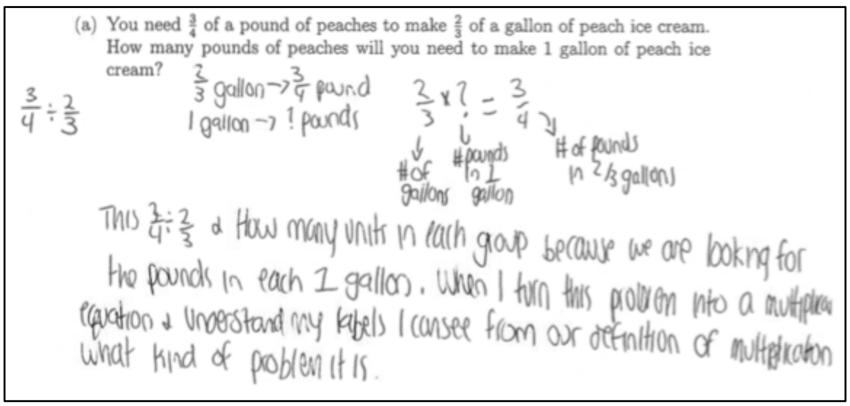

Figure 2. Kelly uses tools from class to identify partitive division (December).

Figure 2 shows Kelly's work on the first word problem. The ratio table in the top left of Kelly's work and the multiplication equation in the top right illustrate the tools from class. In particular, the course instruction used multiplication with an unknown factor, represented in Kelly's work with a question mark, to represent division situations. The units Kelly attached to the $^{2}/_{3}$, the question mark, and the $^{3}/_{4}$ also followed course instruction closely. For reasons that are not clear, Kelly answered part (b) of the same question incorrectly. The only difference was the question, which asked, "How many gallons of peach ice cream can you make from 1 pound of peaches?" Although this is also partitive division, Kelly set it up as a measurement division problem and got confused. The final exam presented six word problems and asked which could be modeled by multiplication, division, or subtraction, again with the fractions $^{2}/_{3}$ and $^{3}/_{4}$. Kelly again used ratio tables and multiplication equations like those shown in Figure 2 to identify appropriately all instances of division (partitive and measurement). She also identified appropriately instances of multiplication but then changed her answers. Thus, by the end of the course, her recognition of appropriate operations was largely, but not completely, accurate.

The DTMR Fractions items that measured appropriateness asked whether situations described in word problems could be modeled by multiplication or division. (The test is secure, so we cannot provide specific items.) Kelly did not explain how she selected her choices to the DTMR items, so we cannot be sure if she used ratio tables or the meaning of multiplication illustrated in Figure 2. Nevertheless, her use of the explicit meaning for multiplication and ratio tables on course assessments as instructed in class suggested the number and operations course played a key role in her improved performance; and the contrast between her performance on the DTMR pretest and posttest reflected genuine improved facility identifying which arithmetic operations can be used to model situations described in word problems.

Shifts in Kelly's Reasoning about Partitioning and Iterating

Kelly's performance on assessments for the numbers and operations course indicated frequent use of common denominators when partitioning quantities. Figure 3 shows one example of her work from a test given in September. This example occurred 2 weeks after partitioning had been introduced for the purpose of generating equivalent fractions. The question presented a number line showing the locations of 0 and $^{2}/_{7}$ and asked for the locations of $^{1}/_{7}$ and $^{1}/_{5}$. Kelly began her explanation by stating: "In order to have equally spaced tick marks we must first find a common denominator with $^{1}/_{7}$ and $^{1}/_{5}$." We did not assign much significance to the uneven spacing of her tick marks.

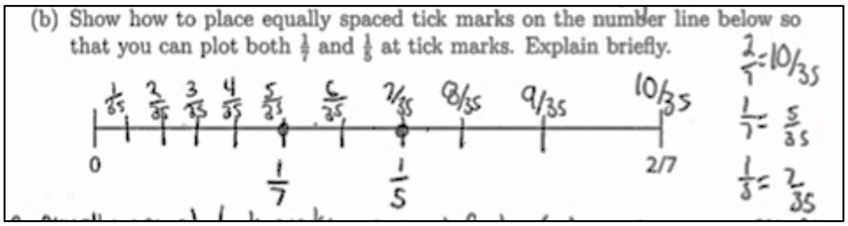

Figure 3. Kelly uses common denominators to partition (September).

Kelly's consistent attention to common denominators served her well in some, but not all, situations: She continued to focus on common denominators as a guide for partitioning when problems called instead for partitioning using common numerators. Partitioning by common numerators is useful for some methods for solving partitive division problems with drawings. This aspect of partitioning was not emphasized in the course. Figure 4 shows Kelly's work on a test given in December, near the end of the course. She generated an appropriate partitive division word problem as asked, again using the meaning for multiplication and the ratio table discussed above. As part of her explanation, she stated "I must find a common denominator of 4" when, in fact, partitioning $^{1}/_{2}$ into two equal parts comes from the least common multiple of the numerators for $^{1}/_{2}$ and $^{2}/_{3}$.

The DTMR Fractions items that measured partitioning and iterating presented a mix of situations calling for partitioning by common denominators and by common numerators. On the DTMR posttest, Kelly demonstrated continued use of partitioning by common denominators but missed nearly all items for which attention to common numerators would be useful. Thus, Kelly's modest increase in performance from the DTMR pretest to posttest reflected her restricted attention to common denominators when partitioning.

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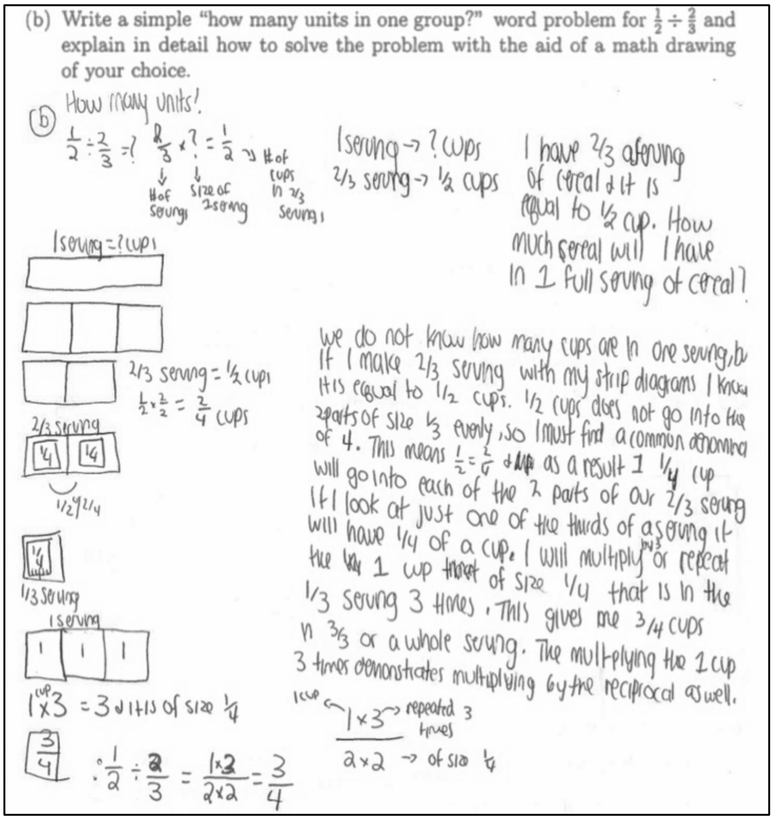

Figure 4. Kelly's reasoning about partitioning and iterating and reversibility (December).

Shifts in Kelly's Reasoning about Referent Unit

Kelly's performance on assessments for the numbers and operations course indicated inconsistent reasoning about referent units. At times she gave clear, correct explanations for the referent units in multiplication and division problems. Figure 5 shows one example where Kelly demonstrated taking part of a part correctly when explaining the meaning of $^{2}/_{3}$ x $^{4}/_{5}$. From her written work we infer that she drew $^{4}/_{5}$ of one whole first and then took $^{2}/_{3}$ of the $^{4}/_{5}$. There were also examples where she assigned appropriate referent units to the quotient in both partitive and measurement division problems. On a test in December, Kelly wrote the following when asked for a measurement division problem that illustrated $^{1}/_{2} \div {}^{2}/_{3}$: "One serving of cereal is $^{2}/_{3}$ cups. How many servings of cereal will I have in $^{1}/_{2}$ cups?" When solving her problem, Kelly was explicit about converting the given fractions into $^{3}/_{6}$ and $^{4}/_{6}$ and using $^{4}/_{6}$ as a new unit.

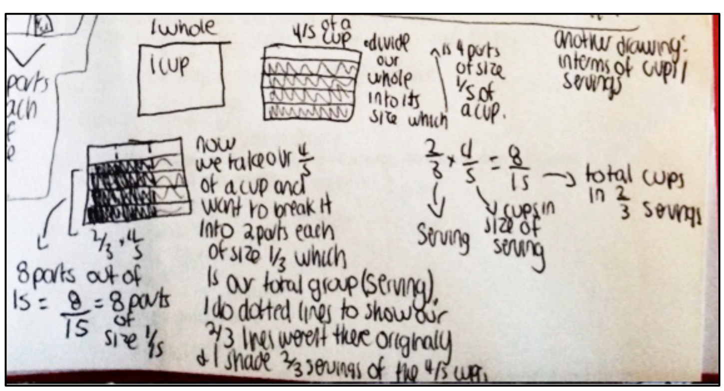

Figure 5. Kelly's reasoning about referent units for fraction multiplication (October).

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At the same time, Kelly's incorrect reasoning about units was evident in several places, including some of her work on the final exam. One problem presented a picture of a 5-part strip with three parts shaded. The problem stated that the entire 5-part strip represented $^7/_2$ acres of land and asked whether, according to the meaning for multiplication developed in the course, the 3 shaded parts represented $^7/_2 * {}^3/_5$ (incorrect) or $^3/_5 * {}^7/_2$ (correct). Ostensibly, this problem was about appropriateness, but a central feature of Kelly's work was incorrect referent units. In particular, she divided each of the 5 parts in half, creating 10 parts, and argued that the shaded region showed $^6/_{10}$, not the $^{21}/_{10}$, which she knew was the product. Thus she conflated $^1/_{10}$ of $^7/_2$ with $^1/_{10}$ of the whole.

The DTMR Fractions items that measured referent units for multiplication and division situations often presented a complete number sentence and several drawings indicating different choices for the referent unit for the product or the quotient. Kelly got some items correct while missing others. Thus her reasoning appeared inconsistent both on course assignments and on the DTMR survey.

Shifts in Kelly's Reasoning about Reversibility

Kelly's performance on written assignments for the numbers and operations course indicated her ability to start with a proper or improper fraction and construct the relevant whole, which is the sense of reversibility used in the DTMR Fractions survey. The examples in Figures 4 and 6 illustrate Kelly's reasoning with reversibility after the Common Core State Standards definition for fraction, which is based on iterating a unit fraction, was introduced in late August. Figure 6 shows Kelly's work on a quiz in September. Given an array of X's that was $^4/_3$ of another array, Kelly partitioned by 4 to find $^1/_3$ of the given array. She explained that "The unit whole of a fraction of $^1/_3$ would be 3 equal parts of each of size $^1/_3$." In the rest or her work (not shown) she then redrew the first three columns of circled X's shown in the figure. In Figure 4 she correctly reasoned that if $^1/_2$ cup makes $^2/_3$ of a serving, then $^1/_4$ cup makes $^1/_3$ of serving. She completed her solution by explaining that since 3 one-thirds make a serving, 3 one-fourth cups must be needed for one serving. On the final exam, she wrote a partitive division word problem for $^1/_3 \div {}^4/_5$ and solved using the same method as the one used for the problem shown in Figure 4. We conjecture that the introduction of the Common Core State Standards early in the course provided Kelly a new perspective on fractions that allowed her to solve reversibility problems quickly.

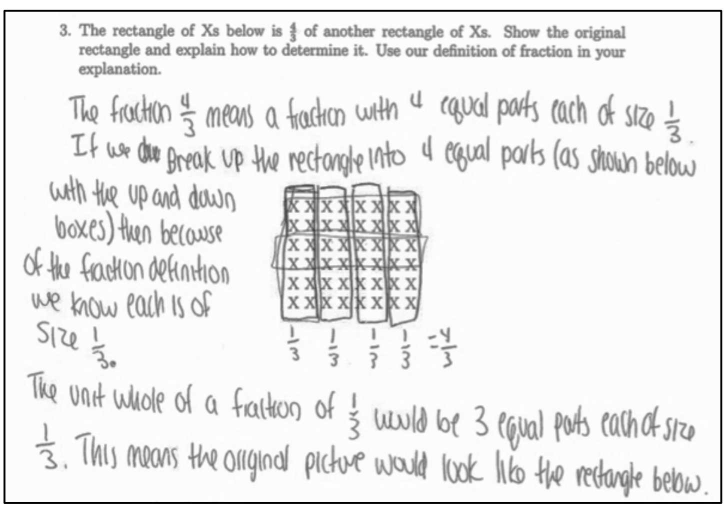

Figure 6. Kelly's reasoning about reversibility (September).

Discussion

The results we present add to an accumulating body of evidence that the DTMR Fractions survey is a valid measure of teachers' reasoning about fraction arithmetic in terms of quantities. Past results

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established the content validity and psychometric properties of the survey. The present study examines the sensitivity of the survey to growth and change in reasoning during a one-semester content course on number and operation. In particular, our results suggest that changes in Kelly's reasoning about fraction arithmetic, as evidenced by her use of tools and definition for fractions introduced during the course, were reasonably consistent with the shift in her performance on the DTMR survey before and after the course, at least in those areas where the content of the survey and course were closely aligned. Further analysis of the remaining dataset is ongoing, and we will present results of that analysis in the near future. The results we present here are a critical next step in developing measures that capture information about moment-to-moment reasoning, narrowing the gap between information about teachers that can be ascertained through large-scale surveys and detailed case studies of problem-solving performance. Narrowing this gap is critical for measuring growth in reasoning necessary for enacting current curriculum standards (e.g., National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010; National Council of Teachers of Mathematics, 2000). Since the approach to measurement we used, based on diagnostic classification models, can be applied to other content areas and to students as well as teachers, the implications for psychometric models as tools for research and practical applications are broad.

References

American Mathematical Society. (2010). *The mathematical education of teachers II*. Conference Board of Mathematical Sciences, Issues in Mathematics Education, Vol. 17. Washington, DC: Author.

Ball, D. L. (1990). Prospective elementary and secondary teachers' understanding of division. *Journal for Research in Mathematics Education*, *21*(2), 132-144.

Bradshaw, L., Izsák, A., Templin, J., & Jacobson, E. (2014). Diagnosing teachers' understandings of rational number: Building a multidimensional test within the diagnostic classification framework. *Educational Measurement: Issues and Practice*, *33*(1), pp. 2–14.

Hill, H. C. (2007). Mathematical knowledge of middle school teachers: Implications for the No Child Left Behind policy initiative. *Educational Evaluation and Policy Analysis*, *29*, 95–114.

Izsák, A. (2008). Mathematical knowledge for teaching fraction multiplication. *Cognition and Instruction*, *26*(1), 95-143.

Izsák, A., Jacobson, E., de Araujo, Z., & Orrill, C. H. (2012) Measuring mathematical knowledge for teaching fractions with drawn quantities. *Journal for Research in Mathematics Education*, *43*(4), 391–427.

Kersting, N., Givvin, K., Sotelo, F., & Stigler, J. (2010). Teachers' analysis of classroom video predicts student learning of mathematics: Further explorations of a novel measure of teacher knowledge. *Journal of Teacher Education*, *61*(1–2), 172–181.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Erlbaum.

National Governors Association Center for Best Practices and Council of Chief State School Officers. (2010). *Common core state standards for mathematics*. Washington, DC: National Governors Association Center for Best Practices, Council of Chief State School Officers.

Sowder, J. T., Armstrong, B., Lamon, S., Simon, M., Sowder, L., & Thompson, A. (1998). Educating teachers to teach multiplicative structures in the middle grades. *Journal of Mathematics Teacher Education*, *1*(2), 127–155.